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# From the Perron–Frobenius Equation to the Fokker–Planck Equation

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We show that for certain classes of deterministic dynamical systems the Perron–Frobenius equation reduces to the Fokker–Planck equation in an appropriate scaling limit. By perturbative expansion in a small time scale parameter, we also derive the equations that are obeyed by the first- and second-order correction terms to the Fokker–Planck limit case. In general, these equations describe non-Gaussian corrections to a Langevin dynamics due to an underlying deterministic chaotic dynamics. For double-symmetric maps, the first-order correction term turns out to satisfy a kind of inhomogeneous Fokker–Planck equation with a source term. For a special example, we are able solve the first- and second-order equations explicitly.

**KEY WORDS:** Perron–Frobenius equation; Fokker–Planck equation; scaling limits; maps of Kaplan–Yorke type; corrections to Gaussian behavior; $\Omega$-expansion.

## 1. INTRODUCTION

An interesting problem of statistical physics is the question of how diffusion processes and Brownian motion can arise from simple deterministic dynamical systems with strong chaotic properties.[1–15] It has become evident that a large phase-space dimension is not necessary to obtain "random behavior" of Brownian motion type. Early work in this direction is due to Billingsley.[1] It has become clear that not only Brownian motion, but also Langevin processes can be generated by smooth low-dimensional mappings in an appropriate scaling limit.[9–15] This concept has turned out to be fruitful; in particular, a spatiotemporal version has recently led to a

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successful description of the probability densities of velocity signals measured in fully developed turbulent flows.[16]

In the simplest case, namely the case of a linear Langevin equation, the relevant class of dynamical systems is given by maps of Kaplan–Yorke type.[17] Indeed, Kaplan–Yorke maps, originally introduced as simple examples where one can study fractal dimensions and Liapunov exponents quite easily, do have an important physical interpretation (besides the "filtered map" interpretation of ref. 18). They can be regarded as the deterministic chaotic analog of a linear Langevin equation, where the Gaussian white noise is replaced by a chaotic dynamics. Thus, in the following we call these general types of mappings "maps of linear Langevin type." For a recent introduction to the subject and also a generalization to nonlinear Langevin equations, leading to "maps of nonlinear Langevin type," see ref. 13.

In general, maps of linear Langevin type generate complicated (non-Gaussian, non-Markovian) stochastic processes. However, one can rigorously prove that this complicated process reduces to the Ornstein–Uhlenbeck process (a Gaussian Markov process) in an appropriate scaling limit.[9,10] The scaling limit means that the time difference $\tau$ between subsequent chaotic "kicks" on the particle approaches zero and that the kick strengths are rescaled by $\sqrt{\tau}$. Decreasing $\tau$, there is a transition scenario from complicated chaotic behavior to Gaussian random behavior.[19]

Of course, for small but finite $\tau$ there are corrections to the Gaussian limit behavior due to the underlying deterministic dynamics. What are the general equations governing these corrections? This is an important question; since every physical system possesses a finite time scale $\tau$, the limit $\tau \to 0$ (leading to Gaussian white noise) is just an idealization.

The idea of this paper is to derive a set of equations that is fulfilled by the corrections to the Gaussian limit behavior in the general case, i.e., for *a priori* arbitrary strongly mixing driving forces. Indeed, we will show how the Fokker–Planck equation arises from a deterministic dynamics in the scaling limit, and derive the general equations governing the next-order corrections to the Fokker–Planck equation for small but finite $\tau$. Our starting point is the Perron–Frobenius equation, describing the conservation of probability for a dynamical system of Langevin type. This functional equation is much too complicated to be solved exactly. However, near the Gaussian limit, we may expand this equation in a perturbative way in the small time scale parameter $\tau^{1/2}$. The resulting equations are still complicated, but nevertheless they are much simpler than the original equation. The expansion method allows us to derive the Fokker–Planck equation directly from the Perron–Frobenius equation, and, perhaps even more important, it provides us with the equations that are obeyed by the next-order correction terms. In the simplest case, the first-order correction term turns out to obey a kind of inhomogeneous Fokker–Planck equation with an external source term.

Our approach bears some similarities to van Kampen's $\Omega$-expansion method.[20] Van Kampen starts with the master equation. He then makes a systematic series development in a small parameter $\Omega^{-1/2}$, where $\Omega$ is usually the volume of the system. Similarly, we start from the Perron–Frobenius equation (representing the balance equation of the probability) and expand in the small time scale parameter $\tau^{1/2}$. The difference is that our method yields the correction terms produced by some underlying deterministic ergodic dynamics, whereas van Kampen starts from a stochastic process that is Markovian and governed by the master equation. Since we do not use the Markov assumption, our resulting equations are much more complicated. Nevertheless, in a certain sense our approach presented in the following sections can be regarded as the analog of van Kampen's $\Omega$-expansion in nonlinear dynamics.

We should remark that for a series expansion to make sense, we implicitly assume that the probability density functions are smooth analytical functions. Indeed, this assumption is well supported by various numerical experiments. We have not found a single example of a map of linear Langevin type driven by a mapping with strong mixing properties where for sufficiently small $\tau$ the corrections to the Gaussian distribution were nonsmooth or fractal. In fact, one numerically observes that all singularities of the density [described by a nontrivial $f(\alpha)$ spectrum] already get lost for "medium" values of $\tau$.[19] It is therefore reasonable to assume that near the Gaussian limit case, for sufficiently small $\tau$, a series expansion makes sense. Starting from this ansatz, the Fokker–Planck equation and the next-order equations are obtained by a systematic power expansion in the time scale parameter $\tau^{1/2}$. While the Fokker–Planck equation is the zeroth-order contribution and exact in the limit $\tau \to 0$, the new message of the present paper is the fact that there are general equations that govern the non-Gaussian corrections in the vicinity of the Fokker–Planck limit case, i.e., the contributions of order $\tau^{1/2}$ and $\tau$. For a special example, we show that these equations can be solved exactly. The probability densities calculated in this way are in perfect agreement with a numerically obtained histogram of iterates, which converges to the invariant density due to ergodicity. The ergodicity of the systems studied here was proved in ref. 11.

On the other hand, if the time scale parameter $\tau$ increases to larger values, one typically observes a critical point $\tau_c$ where the probability density of the velocity of the particle suddenly becomes nondifferentiable. At this point the perturbation theory presented here breaks down, and we have a kind of phase transition point where complicated nonanalytical behavior sets in. The critical value $\tau_c$ where this happens is different from mapping to mapping, and can be regarded as a further interesting characteristic quantity associated with each chaotic mapping. For example, for kicks generated by the Ulam map we numerically estimate $\tau_c \approx 0.20$, for the third-order Tschebyscheff map we estimate $\tau_c \approx 0.47$. Whatever map is chosen, the perturbative expansion described in the following sections makes sense for $\tau < \tau_c$ only, and to have good convergence one should even require $\tau \ll \tau_c$.

This paper is organized as follows: In Section 2 we expand the Perron–Frobenius equation of maps of linear Langevin type in the small time scale parameter $\tau^{1/2}$. In Sections 3 and 4 we derive the Fokker–Planck equation and the general equations that govern the non-Gaussian corrections of the probability density of the velocity of the particle. As an example where the functional equations can be solved explicitly, in Section 5 we calculate non-Gaussian corrections for kicks generated by the Ulam map.

## 2. PERTURBATIVE EXPANSION OF THE PERRON–FROBENIUS EQUATION

Consider a dynamical system of linear Langevin type

$$f\colon\quad \begin{aligned} x_{n+1} &= T(x_n) \\ y_{n+1} &= \lambda y_n + \tau^{1/2} x_n \end{aligned} \tag{1}$$

Here $\lambda \in (0, 1)$ and $\tau > 0$ are parameters, and $T\colon X \to X$ is some $\varphi$-mixing mapping.[9] The map $f$ is obtained by integration from the following deterministic chaotic analog of a Langevin equation:

$$\dot{Y} = -\gamma Y + \tau^{1/2} \sum_{n=1}^{\infty} x_{n-1} \delta(t - n\tau) \tag{2}$$

$$x_{n+1} = T(x_n) \tag{3}$$

$T$ determines the time evolution of the kicks. $Y(t)$ can be regarded as the velocity of a kicked damped particle. Equation (1) describes the stroboscopic time evolution of $y_n := Y(n\tau + 0)$. The damping constant $\gamma > 0$ and the time difference $\tau$ between kicks are related to the parameter $\lambda$ by $\lambda = e^{-\gamma\tau}$. For convenience, we will deal with a one-dimensional phase space $X$, although a similar analysis can be performed for higher-dimensional cases. The inverse map is given by

$$f^{-1}\colon\quad \begin{aligned} x_n &= T^{-1}(x_{n+1}) \\ y_n &= \lambda^{-1}(y_{n+1} - \tau^{1/2} T^{-1}(x_{n+1})) \end{aligned} \tag{4}$$

In general, there are several preimages $T^{-1}$. The absolute value of the Jacobi determinant of $f$ is

$$|\det Df| = \lambda\, |T'(x)| \tag{5}$$

The Perron–Frobenius equation

$$\rho_{n+1}(x', y') = \sum_{(x,y)\in f^{-1}(x',y')} \frac{\rho_n(x, y)}{\lambda\, |T'(x)|} \tag{6}$$

governs the time evolution of probability densities $\rho_n(x, y)$; it can be written as a sum over the preimages of $T$:

$$\lambda \rho_{n+1}(x', y) = \sum_{x\in T^{-1}(x')} \frac{1}{|T'(x)|}\, \rho_n(x, \lambda^{-1}(y - \tau^{1/2}x)) \tag{7}$$

Equation (7) with $\lambda = e^{-\gamma\tau}$ is the starting point of our consideration. We will expand it with respect to the parameter $\tau^{1/2}$, which is supposed to be small. In our perturbative analysis we will include all terms up to fourth order in $\tau^{1/2}$. To simplify the notation, we will not explicitly write down any rest term of $O(\tau^{5/2})$, but suppress it in all equations. Without restriction of generality we set $\gamma = 1$, obtaining up to fourth order in $\tau^{1/2}$

$$\lambda = e^{-\tau} = 1 - \tau + \tfrac{1}{2}\tau^2 \tag{8}$$

Since

$$\lambda^{-1}(y - \tau^{1/2}x) = (1 + \tau + \tfrac{1}{2}\tau^2)(y - \tau^{1/2}x) \tag{9}$$

$$= y - \tau^{1/2}x + \tau y - \tau^{3/2}x + \tfrac{1}{2}\tau^2 y \tag{10}$$

we obtain by Taylor expansion

$$\begin{aligned}
&\rho_n(x, \lambda^{-1}(y - \tau^{1/2}x))\\
&\quad= \rho_n(x, y) + \left(-\tau^{1/2}x + \tau y - \tau^{3/2}x + \frac{1}{2}\tau^2 y\right)\frac{\partial}{\partial y}\rho_n(x, y)\\
&\qquad+ \frac{1}{2}(\tau x^2 + \tau^2 y^2 - 2\tau^{3/2}xy + 2\tau^2 x^2)\frac{\partial^2}{\partial y^2}\rho_n(x, y)\\
&\qquad+ \frac{1}{6}(-\tau^{3/2}x^3 + 3\tau^2 x^2 y)\frac{\partial^3}{\partial y^3}\rho_n(x, y) + \frac{1}{24}\tau^2 x^4 \frac{\partial^4}{\partial y^4}\rho_n(x, y)
\end{aligned}$$

$$= \rho_n(x, y) + \tau^{1/2} \left\{ -x \frac{\partial}{\partial y} \rho_n(x, y) \right\}$$

$$+ \tau \left\{ y \frac{\partial}{\partial y} \rho_n(x, y) + \frac{1}{2} x^2 \frac{\partial^2}{\partial y^2} \rho_n(x, y) \right\}$$

$$+ \tau^{3/2} \left\{ -x \frac{\partial}{\partial y} \rho_n(x, y) - xy \frac{\partial^2}{\partial y^2} \rho_n(x, y) - \frac{1}{6} x^3 \frac{\partial^3}{\partial y^3} \rho_n(x, y) \right\}$$

$$+ \tau^2 \left\{ \frac{1}{2} y \frac{\partial}{\partial y} \rho_n(x, y) + \frac{1}{2} y^2 \frac{\partial^2}{\partial y^2} \rho_n(x, y) + x^2 \frac{\partial^2}{\partial y^2} \rho_n(x, y) \right.$$

$$\left. + \frac{1}{2} x^2 y \frac{\partial^3}{\partial y^3} \rho_n(x, y) + \frac{1}{24} x^4 \frac{\partial^4}{\partial y^4} \rho_n(x, y) \right\} \tag{11}$$

Let us introduce a continuous-time suspension $\rho(x, y, t)$ defined by

$$\rho_n(x, y) = \rho(x, y, t) \qquad (t = n\tau) \tag{12}$$

Since we keep $\tau$ small but finite, there are infinitely many such smooth functions $\rho(x, y, t)$. At the present stage, we need not fix the function $\rho(x, y, t)$ for times $t$ other than $n\tau$. The following considerations are valid for *any* smooth suspension $\rho(x, y, t)$ satisfying Eq. (12). A Taylor expansion yields

$$\rho_{n+1}(x, y) = \rho(x, y, n\tau + \tau)$$

$$= \rho(x, y, n\tau) + \tau \frac{\partial}{\partial t} \rho(x, y, n\tau) + \frac{1}{2} \tau^2 \frac{\partial^2}{\partial t^2} \rho(x, y, n\tau)$$

Hence

$$\lambda \rho_{n+1}(x', y)$$

$$= \left(1 - \tau + \frac{1}{2} \tau^2\right) \left\{ \rho(x', y, t) + \tau \frac{\partial}{\partial t} \rho(x', y, t) + \frac{1}{2} \tau^2 \frac{\partial^2}{\partial t^2} \rho(x', y, t) \right\}$$

$$= \rho(x', y, t) + \tau \left\{ -\rho(x', y, t) + \frac{\partial}{\partial t} \rho(x', y, t) \right\}$$

$$+ \tau^2 \left\{ \frac{1}{2} \rho(x', y, t) - \frac{\partial}{\partial t} \rho(x', y, t) + \frac{1}{2} \frac{\partial^2}{\partial t^2} \rho(x', y, t) \right\} \qquad (t = n\tau) \tag{13}$$

The function $\rho(x, y, t)$ still depends on $\lambda$ and thus on $\tau^{1/2}$, since for stroboscopic times $t = n\tau$ it is the density $\rho_n(x, y)$ of the map (1) which explicitly depends on $\lambda$. Inspired by van Kampen's $\Omega$-expansion[20] and the results of ref. 12, we now make the following ansatz:

$$\rho(x, y, t) = \varphi(x, y, t) + \tau^{1/2} a(x, y, t) + \tau b(x, y, t) + \tau^{3/2} c(x, y, t) + \tau^2 d(x, y, t) \tag{14}$$

Here $\varphi$, $a$, $b$, $c$, and $d$ are $\tau$-independent functions. This ansatz is the fundamental assumption of this paper. It says that we assume that among the infinite set of smooth suspensions $\rho(x, y, t)$ there is at least one that satisfies Eq. (14) for *arbitrary* $\tau$ in the vicinity of 0. This includes the limit $\tau \to 0$, where a continuum is approached, and thus we actually expect that the validity of Eq. (14) for arbitrary $\tau$ fixes unique smooth functions $\varphi, a, b, c, \ldots$. On the other hand, the ultimate goal of this paper is to calculate finite-$\tau$ corrections, and for finite $\tau$ we need the functions $\varphi, a, b, c, \ldots$ for stroboscopic times only.

Putting the ansatz (14) into Eqs. (11) and (13) and comparing different powers of $\tau^{1/2}$, one finally obtains the following five coupled functional equations for $\varphi$, $a$, $b$, $c$, and $d$:

$O(\tau^0)$:

$$\varphi(x', y, t) = \sum_{x \in T^{-1}(x')} \frac{1}{|T'(x)|} \varphi(x, y, t) \tag{15}$$

$O(\tau^{1/2})$:

$$a(x', y, t) = \sum_{x \in T^{-1}(x')} \frac{1}{|T'(x)|} \left\{ a(x, y, t) - x \frac{\partial}{\partial y} \varphi(x, y, t) \right\} \tag{16}$$

$O(\tau^1)$:

$$\begin{aligned} b(x', y, t) - \varphi(x', y, t) + \frac{\partial}{\partial t} \varphi(x', y, t) &= \sum_{x \in T^{-1}(x')} \frac{1}{|T'(x)|} \left\{ b(x, y, t) - x \frac{\partial}{\partial y} a(x, y, t) + y \frac{\partial}{\partial y} \varphi(x, y, t) \right. \\ &\quad \left. + \frac{1}{2} x^2 \frac{\partial^2}{\partial y^2} \varphi(x, y, t) \right\} \end{aligned} \tag{17}$$

$O(\tau^{3/2})$:

$$c(x', y, t) - a(x', y, t) + \frac{\partial}{\partial t} a(x', y, t)$$

$$= \sum_{x \in T^{-1}(x')} \frac{1}{|T'(x)|} \left\{ c(x, y, t) - x \frac{\partial}{\partial y} b(x, y, t) + y \frac{\partial}{\partial y} a(x, y, t) \right.$$

$$+ \frac{1}{2} x^2 \frac{\partial^2}{\partial y^2} a(x, y, t) - x \frac{\partial}{\partial y} \varphi(x, y, t) - xy \frac{\partial^2}{\partial y^2} \varphi(x, y, t)$$

$$\left. - \frac{1}{6} x^3 \frac{\partial^3}{\partial y^3} \varphi(x, y, t) \right\} \tag{18}$$

$O(\tau^2)$:

$$d(x', y, t) - b(x', y, t) + \frac{\partial}{\partial t} b(x', y, t) + \frac{1}{2} \varphi(x', y, t)$$

$$- \frac{\partial}{\partial t} \varphi(x', y, t) + \frac{1}{2} \frac{\partial^2}{\partial t^2} \varphi(x', y, t)$$

$$= \sum_{x \in T^{-1}(x')} \frac{1}{|T'(x)|} \left\{ d(x, y, t) - x \frac{\partial}{\partial y} c(x, y, t) + y \frac{\partial}{\partial y} b(x, y, t) \right.$$

$$+ \frac{1}{2} x^2 \frac{\partial^2}{\partial y^2} b(x, y, t) - x \frac{\partial}{\partial y} a(x, y, t)$$

$$- xy \frac{\partial^2}{\partial y^2} a(x, y, t) - \frac{1}{6} x^3 \frac{\partial^3}{\partial y^3} a(x, y, t)$$

$$+ \frac{1}{2} y \frac{\partial}{\partial y} \varphi(x, y, t) + \frac{1}{2} y^2 \frac{\partial^2}{\partial y^2} \varphi(x, y, t) + x^2 \frac{\partial^2}{\partial y^2} \varphi(x, y, t)$$

$$\left. + \frac{1}{2} x^2 y \frac{\partial^3}{\partial y^3} \varphi(x, y, t) + \frac{1}{24} x^4 \frac{\partial^4}{\partial y^4} \varphi(x, y, t) \right\} \tag{19}$$

Substituting Eq. (15) into Eq. (17), we obtain

$$b(x', y, t) = \sum_{x \in T^{-1}(x')} \frac{1}{|T'(x)|} \left\{ b(x, y, t) - x \frac{\partial}{\partial y} a(x, y, t) \right.$$

$$\left. + \frac{\partial}{\partial y} (y \varphi(x, y, t)) + \frac{1}{2} x^2 \frac{\partial^2}{\partial y^2} \varphi(x, y, t) - \frac{\partial}{\partial t} \varphi(x, y, t) \right\} \tag{20}$$

Notice that the last term on the right-hand side of Eq. (20) is a Fokker–Planck operator with *variable* variance $x^2$. Equation (20) is a kind of combination of a Perron–Frobenius equation for the map $T$ with a Fokker–Planck equation, where the variance is not constant, but given by $x^2$.

Similarly, putting Eq. (16) into Eq. (18) we get

$$\begin{aligned} c(x', y, t) = \sum_{x \in T^{-1}(x')} \frac{1}{|T'(x)|} \Big\{ & c(x, y, t) - x \frac{\partial}{\partial y} b(x, y, t) \\ & + \frac{\partial}{\partial y} (y a(x, y, t)) + \frac{1}{2} x^2 \frac{\partial^2}{\partial y^2} a(x, y, t) - \frac{\partial}{\partial t} a(x, y, t) \\ & - 2x \frac{\partial}{\partial y} \varphi(x, y, t) + x \frac{\partial}{\partial y} \frac{\partial}{\partial t} \varphi(x, y, t) - xy \frac{\partial^2}{\partial y^2} \varphi(x, y, t) \\ & - \frac{1}{6} x^3 \frac{\partial^3}{\partial y^3} \varphi(x, y, t) \Big\} \end{aligned} \tag{21}$$

Finally, putting Eqs. (15) and (20) into Eq. (19), we obtain

$$\begin{aligned} d(x', y, t) = \sum_{x \in T^{-1}(x')} \frac{1}{|T'(x)|} \Big\{ & d(x, y, t) - x \frac{\partial}{\partial y} c(x, y, t) + \frac{\partial}{\partial y} (y b(x, y, t)) \\ & + \frac{1}{2} x^2 \frac{\partial^2}{\partial y^2} b(x, y, t) - \frac{\partial}{\partial t} b(x, y, t) \\ & + x \left[ -2 + \frac{\partial}{\partial t} - y \frac{\partial}{\partial y} - \frac{1}{6} x^2 \frac{\partial^2}{\partial y^2} \right] \frac{\partial}{\partial y} a(x, y, t) \\ & + \frac{\partial}{\partial y} (y \varphi(x, y, t)) + \frac{1}{2} x^2 \frac{\partial^2}{\partial y^2} \varphi(x, y, t) - \frac{\partial}{\partial t} \varphi(x, y, t) \\ & - \frac{\partial}{\partial t} \left[ \frac{\partial}{\partial y} (y \varphi(x, y, t)) + \frac{1}{2} x^2 \frac{\partial^2}{\partial y^2} \varphi(x, y, t) - \frac{\partial}{\partial t} \varphi(x, y, t) \right] \\ & + \frac{1}{2} y \frac{\partial}{\partial y} \varphi(x, y, t) + \frac{1}{2} y^2 \frac{\partial^2}{\partial y^2} \varphi(x, y, t) \\ & + x^2 \left[ 1 + \frac{1}{2} y \frac{\partial}{\partial y} + \frac{1}{24} x^2 \frac{\partial^2}{\partial y^2} \right] \frac{\partial^2}{\partial y^2} \varphi(x, y, t) \\ & - \frac{1}{2} \varphi(x, y, t) + \frac{\partial}{\partial t} \varphi(x, y, t) - \frac{1}{2} \frac{\partial^2}{\partial t^2} \varphi(x, y, t) \Big\} \end{aligned} \tag{22}$$

## 3. INTEGRATION OVER $x$

The previous equations are coupled functional equations connecting the functions $\varphi$, $a$, $b$, $c$, and $d$. These functions contain information on the density of the entire 2-dimensional (or even higher-dimensional) phase space, i.e., they depend on the tuple $(x, y)$. In many cases, however, we are just interested in the marginal distribution

$$p(y, t) = \int dx\, \rho(x, y, t) \tag{23}$$

of the $y$ variable, obtained by integration over all possible $x$ values. In the physical picture, $p(y, t)$ corresponds to the velocity distribution of the particle, whereas $\rho(x, y, t)$ has no direct physical interpretation. The integration over $x$ indeed yields a simplification, transforming the coupled functional equations into coupled differential equations.

Equations (15)–(22) are equations of the general form

$$f(x', y, t) = \sum_{x \in T^{-1}(x')} \frac{1}{|T'(x)|}\, g(x, y, t) \tag{24}$$

where $f$ and $g$ are appropriate functions. Without restriction of generality, let us assume that the phase space of the mapping $T$ is the interval $X = [-1, 1]$. The marginal functions, obtained by integration over all $x$ values, are denoted by

$$\tilde{f}(y, t) := \int_{-1}^{1} dx\, f(x, y, t) \tag{25}$$

$$\tilde{g}(y, t) := \int_{-1}^{1} dx\, g(x, y, t) \tag{26}$$

For complete maps, i.e., maps $T$ that are piecewise monotonous on intervals $[a_i, a_{i+1}]$ and that satisfy $T(a_i) = 1$ or $T(a_i) = -1$ for all $a_i$, one can easily prove the following.

**Integration Lemma.** Let $T$ be a complete map and let $f(x, y, t)$ and $g(x, y, t)$ be two functions satisfying Eq. (24). Then the marginal functions coincide:

$$\tilde{f}(y, t) = \tilde{g}(y, t) \tag{27}$$

The proof is trivial and just relates to an elementary property of the Perron–Frobenius operator.

Let us now apply the integration lemma to Eqs. (15)–(22). We denote the marginal functions as follows:

$$p_0(y, t) = \int dx\, \varphi(x, y, t) \tag{28}$$

$$\alpha(y, t) = \int dx\, a(x, y, t) \tag{29}$$

$$\beta(y, t) = \int dx\, b(x, y, t) \tag{30}$$

$$\gamma(y, t) = \int dx\, c(x, y, t) \tag{31}$$

$$\delta(y, t) = \int dx\, d(x, y, t) \tag{32}$$

Applying the integration lemma to Eq. (15) yields just the trivial statement $p_0(y, t) = p_0(y, t)$. But from the higher-order equations we get nontrivial statements. Equation (16) yields

$$\frac{\partial}{\partial y} \int dx\, x\varphi(x, y, t) = 0 \tag{33}$$

Equation (20) implies

$$\frac{\partial}{\partial y} \int dx\, xa(x, y, t) = \frac{\partial}{\partial y} (yp_0(y, t)) + \frac{1}{2} \frac{\partial^2}{\partial y^2} \int dx\, x^2 \varphi(x, y, t) - \frac{\partial}{\partial t} p_0(y, t) \tag{34}$$

Equation (21) gives

$$\begin{aligned} \frac{\partial}{\partial y} \int dx\, xb(x, y, t) = {} & \frac{\partial}{\partial y} (y\alpha(y, t)) + \frac{1}{2} \frac{\partial^2}{\partial y^2} \int dx\, x^2 a(x, y, t) - \frac{\partial}{\partial t} \alpha(y, t) \\ & + \int dx\, x \left\{ -2 + \frac{\partial}{\partial t} - y \frac{\partial}{\partial y} - \frac{1}{6} x^2 \frac{\partial^2}{\partial y^2} \right\} \frac{\partial}{\partial y} \varphi(x, y, t) \end{aligned} \tag{35}$$

Finally, Eq. (22) yields

$$\begin{aligned} \frac{\partial}{\partial y} \int dx\, xc(x, y, t) = {} & \frac{\partial}{\partial y} (y\beta(y, t)) + \frac{1}{2} \frac{\partial^2}{\partial y^2} \int dx\, x^2 b(x, y, t) - \frac{\partial}{\partial t} \beta(y, t) \\ & - \int dx\, x \left\{ 2 - \frac{\partial}{\partial t} + y \frac{\partial}{\partial y} + \frac{1}{6} x^2 \frac{\partial^2}{\partial y^2} \right\} \frac{\partial}{\partial y} a(x, y, t) \end{aligned}$$

$$+\left(1-\frac{\partial}{\partial t}\right)\left\{\frac{\partial}{\partial y}(yp_0(y,t))\right.$$

$$\left.+\frac{1}{2}\frac{\partial^2}{\partial y^2}\int dx\, x^2\varphi(x,y,t)-\frac{\partial}{\partial t}p_0(y,t)\right\}$$

$$+\left\{\frac{1}{2}y\frac{\partial}{\partial y}+\frac{1}{2}y^2\frac{\partial^2}{\partial y^2}-\frac{1}{2}+\frac{\partial}{\partial t}-\frac{1}{2}\frac{\partial^2}{\partial t^2}\right\}p_0(y,t)$$

$$+\left(\frac{\partial^2}{\partial y^2}+\frac{1}{2}y\frac{\partial^3}{\partial y^3}\right)\int dx\, x^2\varphi(x,y,t)$$

$$+\frac{1}{24}\frac{\partial^4}{\partial y^4}\int dx\, x^4\varphi(x,y,t) \tag{36}$$

Let us first deal with the zeroth-order term $\varphi(x, y, t)$. Equation (15) is solved by any function $\varphi(x, y, t)$ of the form

$$\varphi(x,y,t)=h(x)\,p_0(y,t) \tag{37}$$

where $h(x)$ is the natural invariant density of the $T$-dynamics. To obtain a compact notation, we will use the symbol $\langle\cdots\rangle$ for expectations with respect to $h(x)$. Equation (33) then implies

$$\frac{\partial}{\partial y}p_0(y,t)\int dx\, xh(x)=\langle x\rangle\frac{\partial}{\partial y}p_0(y,t)=0 \tag{38}$$

In case that $p_0(y, t) \neq \text{const}_y$ this means $\langle x\rangle = 0$, which shows that the ansatz (14) makes sense for maps with vanishing average only. Equation (34) becomes

$$\frac{\partial}{\partial y}(yp_0(y,t))+\frac{1}{2}\langle x^2\rangle\frac{\partial^2}{\partial y^2}p_0(y,t)-\frac{\partial}{\partial t}p_0(y,t)=\frac{\partial}{\partial y}\int dx\, xa(x,y,t) \tag{39}$$

This is a kind of inhomogeneous Fokker–Planck equation with a source term $(\partial/\partial y)\int dx\, xa(x, y, t)$. It reduces to a Fokker–Planck equation for the case that $(\partial/\partial y)\int dx\, xa(x, y, t) = 0$. For the case that $(\partial/\partial y)\int dx\, a(x, y, t)$ is proportional to $(\partial^2/\partial y^2)\, p_0(y, t)$, we also obtain a Fokker–Planck equation, but with a different diffusion constant. Hence, in order to determine $p_0(y, t)$, we have to determine $\int dx\, xa(x, y, t)$ with the help of Eq. (16).

## 4. SIMPLIFICATION FOR DOUBLE SYMMETRIC MAPS

We now consider a subclass of maps $T$ determined by the property that both the mapping $T$ as well as its natural invariant density $h$ are symmetric:

$$T(x) = T(-x) \tag{40}$$

$$h(x) = h(-x) \tag{41}$$

We call these types of maps "double symmetric." Examples are maps conjugated to even Tschebyscheff polynomials. For double-symmetric maps we have

$$\sum_{x \in T^{-1}(x')} \frac{1}{|T'(x)|} xh(x) \frac{\partial}{\partial y} p_0(y, t) = 0 \tag{42}$$

since for each $x$, $-x$ is also a preimage. Hence Eq. (16) reduces to

$$a(x', y, t) = \sum_{x \in T^{-1}(x')} \frac{1}{|T'(x)|} a(x, y, t) \tag{43}$$

Notice that this is just the same equation as the one satisfied by $\varphi$. It is solved by a function of the form

$$a(x, y, t) = h(x)\, \alpha(y, t) \tag{44}$$

Moreover, we obtain

$$b(x', y, t) = \sum_{x \in T^{-1}(x')} \frac{1}{|T'(x)|} \left\{ b(x, y, t) + h(x) \left[ \frac{\partial}{\partial y} (y p_0(y, t)) + \frac{1}{2} x^2 \frac{\partial^2}{\partial y^2} p_0(y, t) - \frac{\partial}{\partial t} p_0(y, t) \right] \right\} \tag{45}$$

$$c(x', y, t) = \sum_{x \in T^{-1}(x')} \frac{1}{|T'(x)|} \left\{ c(x, y, t) - x \frac{\partial}{\partial y} b(x, y, t) + h(x) \left[ \frac{\partial}{\partial y} (y \alpha(y, t)) + \frac{1}{2} x^2 \frac{\partial^2}{\partial y^2} \alpha(y, t) - \frac{\partial}{\partial t} \alpha(y, t) \right] \right\} \tag{46}$$

$$d(x', y, t) = \sum_{x \in T^{-1}(x')} \frac{1}{|T'(x)|} \left\{ d(x, y, t) - x \frac{\partial}{\partial y} c(x, y, t) + \frac{\partial}{\partial y} (y b(x, y, t)) + \frac{1}{2} x^2 \frac{\partial^2}{\partial y^2} b(x, y, t) - \frac{\partial}{\partial t} b(x, y, t) \right.$$

$$+h(x)\left[\left(1-\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial y}(yp_0(y,t))\right.\right.$$
$$\left.+\frac{1}{2}x^2\frac{\partial^2}{\partial y^2}p_0(y,t)-\frac{\partial}{\partial t}p_0(y,t)\right)$$
$$+\frac{1}{2}y\frac{\partial}{\partial y}p_0(y,t)+\frac{1}{2}y^2\frac{\partial^2}{\partial y^2}p_0(y,t)$$
$$+x^2\left(1+\frac{1}{2}y\frac{\partial}{\partial y}+\frac{1}{24}x^2\frac{\partial^2}{\partial y^2}\right)\frac{\partial^2}{\partial y^2}p_0(y,t)$$
$$\left.\left.-\frac{1}{2}p_0(y,t)+\frac{\partial}{\partial t}p_0(y,t)-\frac{1}{2}\frac{\partial^2}{\partial t^2}p_0(y,t)\right]\right\} \quad (47)$$

The integrated equations reduce to

$$0=\frac{\partial}{\partial y}(yp_0(y,t))+\frac{1}{2}\langle x^2\rangle\frac{\partial^2}{\partial y^2}p_0(y,t)-\frac{\partial}{\partial t}p_0(y,t) \quad (48)$$

$$\frac{\partial}{\partial y}\int dx\, xb(x,y,t)=\frac{\partial}{\partial y}(y\alpha(y,t))+\frac{1}{2}\langle x^2\rangle\frac{\partial^2}{\partial y^2}\alpha(y,t)-\frac{\partial}{\partial t}\alpha(y,t) \quad (49)$$

$$\frac{\partial}{\partial y}\int dx\, xc(x,y,t)=\frac{\partial}{\partial y}(y\beta(y,t))+\frac{1}{2}\frac{\partial^2}{\partial y^2}\int dx\, x^2b(x,y,t)-\frac{\partial}{\partial t}\beta(y,t)$$
$$+\left\{\frac{1}{2}y\frac{\partial}{\partial y}+\frac{1}{2}y^2\frac{\partial^2}{\partial y^2}+\langle x^2\rangle\frac{\partial^2}{\partial y^2}+\frac{1}{2}\langle x^2\rangle y\frac{\partial^3}{\partial y^3}\right.$$
$$\left.+\frac{1}{24}\langle x^4\rangle\frac{\partial^4}{\partial y^4}-\frac{1}{2}+\frac{\partial}{\partial t}-\frac{1}{2}\frac{\partial^2}{\partial t^2}\right\}p_0(y,t) \quad (50)$$

Notice that Eq. (48) is the Fokker–Planck equation. We obtain the result that for double-symmetric maps the diffusion constant is always given by $\langle x^2\rangle$. Equation (49) is of the same type as Eq. (39); it is an inhomogeneous Fokker–Planck equation, but now—due to the condition of double symmetry—it is satisfied by $\alpha(y,t)$ rather than $p_0(y,t)$.

Subtracting the zero given by Eq. (48) from the right-hand side of Eq. (45), we may also write Eq. (45) as

$$b(x',y,t)=\sum_{x\in T^{-1}(x')}\frac{1}{|T'(x)|}$$
$$\times\left\{b(x,y,t)+h(x)\frac{1}{2}(x^2-\langle x^2\rangle)\frac{\partial^2}{\partial y^2}p_0(y,t)\right\} \quad (51)$$

Similarly, Eq. (46) can be written as

$$c(x', y, t) = \sum_{x \in T^{-1}(x')} \frac{1}{|T'(x)|} \left\{ c(x, y, t) - x \frac{\partial}{\partial y} b(x, y, t) \right.$$
$$\left. + h(x) \left[ \frac{\partial}{\partial y} \int dx\, xb(x, y, t) + \frac{1}{2} (x^2 - \langle x^2 \rangle) \frac{\partial^2}{\partial y^2} \alpha(y, t) \right] \right\} \quad (52)$$

## 5. EXPLICIT SOLUTION FOR $T(x) = 1 - 2x^2$

Although the functional equations of the previous sections look quite complicated, it is remarkable that for certain mappings $T$ an explicit solution can be found. An example is the Ulam map $T(x) = 1 - 2x^2$. Here we have $\langle x^2 \rangle = 1/2$, and the stationary solution of the Fokker–Planck equation (48) is given by

$$p_0(y, t) = \left( \frac{2}{\pi} \right)^{1/2} e^{-2y^2} \quad (53)$$

To obtain the next-order correction term $\alpha(y, t)$, we first have to determine the inhomogeneous source term $(\partial/\partial y) \int dx\, xb(x, y, t)$ in Eq. (49) by solving Eq. (51). Let us choose the following ansatz for the solution of Eq. (51):

$$b(x, y, t) = h(x)\, \beta_0(y, t) + xh(x)\, \beta_1(y, t) \quad (54)$$

Here $\beta_0$ and $\beta_1$ are appropriate functions independent of $x$. Putting Eq. (54) into Eq. (51), we obtain on the left-hand side

$$l = h(x')\, \beta_0(y, t) + x'h(x')\, \beta_1(y, t)$$
$$= h(T(x))[\beta_0(y, t) + (1 - 2x^2)\, \beta_1(y, t)] \quad (55)$$

and on the right-hand side

$$r = \sum_{x \in T^{-1}(x')} \frac{1}{|T'(x)|} \left[ h(x)\, \beta_0(y, t) + xh(x)\, \beta_1(y, t) \right.$$
$$\left. + h(x) \frac{1}{2} \left( x^2 - \frac{1}{2} \right) \frac{\partial^2}{\partial y^2} p_0(y, t) \right]$$
$$= \sum_{x \in T^{-1}(x')} \frac{h(x)}{|T'(x)|} \left[ \beta_0(y, t) - \frac{1}{4} (1 - 2x^2) \frac{\partial^2}{\partial y^2} p_0(y, t) \right] \quad (56)$$

From $l=r$ we get $\beta_0(y,t)$ arbitrary and

$$\beta_1(y,t) = -\frac{1}{4}\frac{\partial^2}{\partial y^2} p_0(y,t) \tag{57}$$

Thus, in the stationary case Eq. (53) yields

$$\beta_1(y,t) = \left(\frac{2}{\pi}\right)^{1/2} (1-4y^2)\, e^{-2y^2} \tag{58}$$

From this we can evaluate the left-hand side of the inhomogeneous Fokker–Planck equation (49):

$$\int dx\, xb(x,y,t) = \langle x^2 \rangle\, \beta_1(y,t) = \frac{1}{2}\left(\frac{2}{\pi}\right)^{1/2} (1-4y^2)\, e^{-2y^2} \tag{59}$$

The stationary solution of Eq. (49) satisfies

$$\frac{\partial}{\partial y}\alpha(y) + 4y\alpha(y) = 2\left(\frac{2}{\pi}\right)^{1/2} (1-4y^2)\, e^{-2y^2} + \text{const} \tag{60}$$

In general, all solutions of the first-order linear differential equation

$$\frac{\partial}{\partial y}\alpha + g(y)\,\alpha + f(y) = 0 \tag{61}$$

are given by

$$\alpha(y) = e^{-G(y)}\left[C - \int_{\mathrm{I}} f(y)\, e^{G(y)}\right] \tag{62}$$

Here $G$ is an indefinite integral of $g$,

$$\frac{\partial}{\partial y} G(y) = g(y) \tag{63}$$

and $\int_{\mathrm{I}} \cdots$ denotes an indefinite integral of the argument. In our case

$$g(y) = 4y \tag{64}$$

$$f(y) = 2\left(\frac{2}{\pi}\right)^{1/2} (4y^2-1)\, e^{-2y^2} - \text{const} \tag{65}$$

This yields

$$G(y) = 2y^2 \tag{66}$$

$$\alpha(y) = e^{-2y^2}\left[C - 2\left(\frac{2}{\pi}\right)^{1/2}\int_1 (4y^2 - 1) + \text{const}\cdot\int_1 e^{2y^2}\right] \tag{67}$$

For $C = \text{const} = 0$ we obtain

$$\alpha(y) = \left(\frac{2}{\pi}\right)^{1/2} e^{-2y^2}\left(2y - \frac{8}{3}y^3\right) \tag{68}$$

To obtain the second-order correction term, we have to solve the coupled system of equations (50) and (52). Putting Eq. (54) into Eq. (52) and again using the symmetry of the preimages of $T$, we can write Eq. (52) as

$$c(x', y, t) = \sum_{x\in T^{-1}(x')} \frac{1}{|T'(x)|}\left\{c(x, y, t) + (1 - 2x^2)\, h(x)\left[\frac{1}{2}\frac{\partial}{\partial y}\beta_1(y, t) - \frac{1}{4}\frac{\partial^2}{\partial y^2}\alpha(y, t)\right]\right\} \tag{69}$$

Similarly as Eq. (51), this equation is solved by the ansatz

$$c(x, y, t) = h(x)\,\gamma_0(y, t) + xh(x)\,\gamma_1(y, t) \tag{70}$$

We obtain $\gamma_0(y, t)$ arbitrary and

$$\gamma_1(y, t) = \frac{1}{2}\frac{\partial}{\partial y}\beta_1(y, t) - \frac{1}{4}\frac{\partial^2}{\partial y^2}\alpha(y, t) \tag{71}$$

Putting Eqs. (70), (71), (54), and (57) into Eq. (50), we arrive at the following equation for the unknown function $\beta_0(y, t) = \beta(y, t)$:

$$\begin{aligned} 0 &= \frac{\partial}{\partial y}(y\beta(y, t)) + \frac{1}{4}\frac{\partial^2}{\partial y^2}\beta(y, t) - \frac{\partial}{\partial t}\beta(y, t) \\ &\quad + \left\{\frac{1}{2}y\frac{\partial}{\partial y} + \frac{1}{2}(y^2 + 1)\frac{\partial^2}{\partial y^2} + \frac{1}{4}y\frac{\partial^3}{\partial y^3} + \frac{5}{64}\frac{\partial^4}{\partial y^4}\right. \\ &\quad \left. - \frac{1}{2} + \frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial t^2}\right\} p_0(y, t) + \frac{1}{8}\frac{\partial^3}{\partial y^3}\alpha(y, t) \end{aligned} \tag{72}$$

In the stationary case, we can use Eqs. (53) and (68) to obtain

$$0=\frac{\partial}{\partial y}(y\beta(y))+\frac{1}{4}\frac{\partial^2}{\partial y^2}\beta(y)+\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{64}{3}y^6-68y^4+46y^2-\frac{15}{4}\right)e^{-2y^2} \tag{73}$$

One can easily check that this differential equation is solved by

$$\beta(y)=\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{32}{9}y^6-\frac{31}{3}y^4+\frac{15}{2}y^2-\frac{37}{48}\right)e^{-2y^2} \tag{74}$$

The final result for the stationary probability density,

$$p(y)=\left(\frac{2}{\pi}\right)^{1/2}\left\{1+\tau^{1/2}\left(-\frac{8}{3}y^3+2y\right)+\tau\left(\frac{32}{9}y^6-\frac{31}{3}y^4+\frac{15}{2}y^2-\frac{37}{48}\right)+O(\tau^{3/2})\right\}e^{-2y^2} \tag{75}$$

is plotted in Fig. 1 for two different values of $\tau^{1/2}$. The coincidence with a numerically obtained histogram of iterates of the $y$ variable of the map (1) is excellent. In particular, the slight asymmetry of the distribution is reproduced correctly.

Our perturbative approach can (in principle) be extended to arbitrarily high orders in $\tau^{1/2}$. It is interesting to notice that the non-Gaussian corrections of order $\tau^{k/2}$, $k=0, 1, 2$, multiplying the Gaussian function in Eq. (75) are of simple polynomial structure. We conjecture that this is also true for the higher orders $k \geqslant 3$. In this case our approach defines an entire set of polynomials $\mathscr{P}_k$ in the sense that the $k$th-order correction term is of the form $\tau^{k/2}(2/\pi)^{1/2}\mathscr{P}_k(y)\exp(-2y^2)$. The first three polynomials $\mathscr{P}_0$, $\mathscr{P}_1$, $\mathscr{P}_2$ are

$$\mathscr{P}_0=1 \tag{76}$$

$$\mathscr{P}_1=-\frac{8}{3}y^3+2y \tag{77}$$

$$\mathscr{P}_2=\frac{32}{9}y^6-\frac{31}{3}y^4+\frac{15}{2}y^2-\frac{37}{48} \tag{78}$$

We would like to remark that Eq. (75) has already been derived in ref. 12; however, it was obtained by a completely different graph-theoretic method. The advantage of the method presented here is that in principle it is applicable to an arbitrary dynamics $T$, rather than just to the special example $T(x)=1-2x^2$.

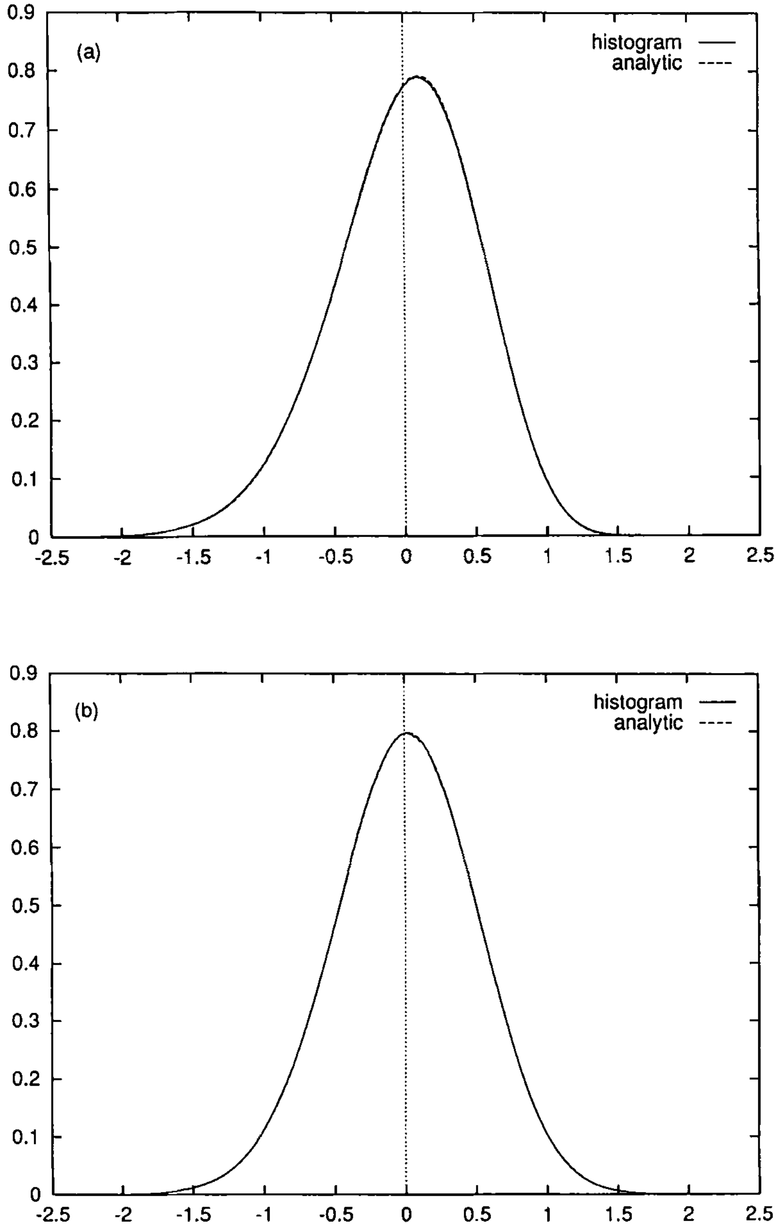


Fig. 1. Histogram of the $y$ variable of the map (1) with $T(x) = 1 - 2x^2$, $\gamma = 1$, and (a) $\tau^{1/2} = 0.20$, (b) $\tau^{1/2} = 0.05$. Also shown is the second-order analytical result, Eq. (75).

We once again notice[12,16] that the Ulam map is distinguished in comparison to other mappings. For this map the first-order correction term $\alpha(y, t)$ obeys a relatively simple differential equation, namely the inhomogeneous Fokker–Planck equation

$$\frac{\partial}{\partial y}(y\alpha(y, t)) + \frac{1}{4}\frac{\partial^2}{\partial y^2}\alpha(y, t) - \frac{\partial}{\partial t}\alpha(y, t) = -\frac{1}{8}\frac{\partial^3}{\partial y^3}p_0(y, t) \tag{79}$$

Moreover, the second-order correction term $\beta(y, t)$ obeys the relatively simple equation (72). In general, for some arbitrary mapping $T$ the correction terms obey the much more complicated equations (34)–(36).

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